‘String theory’ in science lessons: the investigation of a notoriously knotty problem

Robert A. J. Matthews

ABSTRACT Conveying the scope and power of the scientific method in an engaging and accessible way is a major challenge in science education. I show how the well-known (and irksome) phenomenon of the knotting of flex, cord and similarly ‘stringy’ objects provides an unusual opportunity for students to gain experience with cutting-edge mathematics, experimental design and data analysis. I describe a nationwide schools project to investigate the phenomenon which led to confirmation of the so-called Loop Conjecture, and give suggestions for further investigations.

Science and its methods can seem remote from everyday experience, especially when media headlines are routinely being made in research involving scales either unimaginably large (for example, the search for cosmic dark matter) or inconceivably small (such as the quest for the Higgs boson). Textbooks have long recognised the value of illustrations of the application of science in everyday life, and are replete with examples such as the centripetal force acting on clothes inside a spin-dryer. Yet many such examples are uninspiring, or oversimplified to the point of being incorrect (for example, the invocation of Bernoulli’s principle to explain the action of aerofoils). In a previous article in *School Science Review* (Matthews, 2001), I suggested that urban myths could serve as a useful stimulus for scientific projects involving the use of scientific principles outside their usual setting. In particular, I showed how Murphy’s Law of Toast – that toast has a propensity to land butter-side down after sliding off a plate – provided the basis for such a project. As a demonstration, I drew on the experiences and results of a UK-wide experiment designed to investigate the truth of the ‘urban myth’ (which, we note in passing, was shown to be correct around 60% of the time).

In this article, I describe the investigation of another such popular belief widely held by users of headphones, garden hoses, climbing ropes, etc. This belief may be termed Murphy’s Law of String, according to which ‘If flex, cord or any similarly stringy object can end up with a knot in it, it will do.’

In common with Murphy’s Law of Toast, this urban myth has several features making it suitable as the basis of a school science project:

- It is a well-known phenomenon, with the Victorian humourist Jerome K. Jerome making a specific reference to it over a century ago (Jerome, 1889), yet one whose surprisingly deep theoretical basis is not immediately apparent. For a review, see Matthews (1999).
- While familiar, the ‘spontaneous’ formation of knots is not a trivial phenomenon. It has potentially life-threatening consequences for climbers, by reducing the breaking strength of ropes by up to 50% (Warner, 1996). The formation of knots in DNA and proteins is also the focus of research in biochemistry (see, for example, Liu *et al.* (2009) and Sułkowska *et al.* (2012));
- Experiments to investigate its reality, causes and potential remedies are simple, safe and inexpensive, requiring little more than lengths of suitable cord.
- Data collection and analysis is relatively straightforward and easily understood.

Theoretical background

Perhaps surprisingly, considering its ubiquity, research into the phenomenon of spontaneous knotting began only around 50 years ago. Observations of knotting in polymer chains led to the so-called Frisch–Wasserman–Delbrück (FWD) Conjecture (Frisch and Wasserman, 1961; Delbrück, 1962), according to which the probability of a randomly agitated string forming a knot tends exponentially rapidly to certainty with increasing string length, so that, for a string of length L,

$$\Pr(\text{no. knots in string of length } L) \sim e^{-\alpha L} \quad \text{where } \alpha > 0 \qquad (1)$$

Again, perhaps surprisingly, a rigorous proof of the FWD Conjecture did not emerge until the late 1980s (Sumners and Whittington, 1988). It is based on a relatively esoteric branch of mathematics known as the theory of self-avoiding walks (SAWs). These are random walks on the three-dimensional lattice constrained by the requirement that the walk does not pass through the same node twice. This constraint makes the theory of SAWs of obvious relevance to 'real-life' knots, whose physical thickness means they cannot pass through themselves, but also leads to considerable technical complexity, as each step in the random walk is potentially affected by previous steps.

The proof of the FWD Conjecture centres on showing that the probability of a SAW going through the steps needed to form the simplest of all common knots, the so-called trefoil (Figure 1), rises exponentially with length.

This is a celebrated result in the theory of SAWs, not least because it is one of relatively few formally proved results in this branch of mathematics, which remains at the frontiers of research. The proof also has clear implications for Murphy's Law of String. It suggests that this 'urban myth' has rigorous foundations, with spontaneous knotting being the result of the string's random exploration of its surrounding space, which can lead to the string going through the moves necessary for the formation of simple knots.

Figure 1 The trefoil knot

That in turn suggests ways of circumventing Murphy's Law of String. The most obvious is to prevent the string from randomly exploring its space; this, in essence, is the purpose of 'cable tidy' devices. However, the proof's demonstration of an exponential dependence on string length suggest that another, simpler, means of reducing knotting risk, based on the Loop Conjecture (Matthews, 2009, 2014). In essence, this states that randomly jumbled string is less likely to become knotted if its ends are joined together. Specifically, the conjecture states that the act of looping the string reduces α, the coefficient of the length dependency in Equation 1, by at least a factor of 2. This follows from the simple observation that connecting the two ends of an open-ended piece of string halves its effective linear length, as is immediately obvious by forming a loop from a given length of string and stretching it out. Expressed mathematically, the Loop Conjecture is then:

$$\Pr(\text{no. knots in looped string, length } L) \sim e^{-\beta L} \quad \text{where } 0 < \beta < \alpha/2 \qquad (2)$$

While mathematically well motivated, Equation 2 may not be valid in real life. Investigating the validity of both Equations 1 and 2 became the basis for the nationwide project I now describe.

Project design

The project had two educational aims. The first was to encourage students to recognise how a 'trivial' phenomenon such as the tangling of headphone flex comes within the remit of the mathematical sciences. The second was to illustrate the classic scientific process of making a prediction, and then testing it using data collection and analysis.

To assist teachers with this, I prepared an information pack (obtainable in electronic format from the author) containing:

- historical background about the phenomenon of spontaneous knotting and its significance;
- practical details of how to plan and conduct the experiments;
- methods of data analysis;
- suggestions for further studies.

The basic experimental procedure was kept simple enough to be carried out with minimal supervision while still providing clear outcomes. According to the theory outlined above, if different lengths of knot-free string are jumbled at random in N trials then the proportion of these trials resulting in at least one knot should increase with increasing length of string according to Equation 1 above. The experimental design calls for various lengths of ordinary parcel string to be jumbled by hand 20 times, to give a reasonable number of data points by which to gauge the proportion of knot-free jumblings, which becomes an estimate of the probability of a knot-free state for a specific length of string. This should then be repeated for at least six different lengths of string to give enough data points to demonstrate whether Equation 1 holds. Testing Equation 2 simply requires repeating the procedure but with each length of string having its ends joined, forming a loop.

A pilot study conducted by the author (Matthews, 2009) indicated that the entire data collection process could be carried out in a class or STEM club environment, and could typically involve a small team of students devoting a total of two to three hours to the experiment.

Table 1 The Coundon Court data set

Length (cm)	Knot-free percentage	
	Open-ended string	Looped string
55	61	100
57	66	99
60	66	88
	51	77
65	45	93
87	65	83
92	52	90
	49	80
95	51	86
	58	82
100	66	83
	58	74
	55	73
115	48	79
	55	78
	56	69
118	46	74
183	45	85
	44	83

In collaboration with the Department of Mathematics at Aston University, a call for participation was put out to UK schools via STEMNET, along with a press release. This resulted in 11 schools expressing interest and requesting the information pack.

Teachers reported that the experiment was simple enough to be performed by relatively young (year 6; age 10–11) students, with the prospect of making a genuine contribution to science seemingly being sufficient to motivate even students with modest mathematical ability.

Results and qualitative analysis

Of the four schools that reported results, Coundon Court school near Coventry completed by far the most comprehensive study, with eight teams from years 6–9 collectively completing over 5000 trials in both the open-ended and closed (looped) states. The size of the Coundon Court data set and the consistency of the equipment and procedures used make it a convenient case study.

The data set covers the outcome of randomly jumbling parcel string of 1.5 mm thickness for a total of 11 different lengths ranging from 55 cm to 183 cm. For each length, data were collected for string in both unlooped and looped states. As identical lengths were sometimes used by different groups, the final data set consists of a total of 19 determinations of the knot-free percentage among the 11 different lengths (Table 1 and Figure 2).

The data in Table 1 and Figure 2 can serve to motivate a classroom discussion about a variety of data collection and analysis issues, according to age and level of scientific sophistication:

- *Do the data provide evidence to support Murphy's Law of String?*
  This states that string in its open-ended form is more likely to become knotted as the length of the string increases. Assessing whether this is true requires interpretation of a graph like that in Figure 2.
- *How clear is the tendency, and what might cause variations from it?*
  This highlights the fact that the data are quite 'noisy', making the tendency somewhat hard to discern from the table or the figure. It also shows the increased clarity that results from putting some kind of 'best-fit' line through data, with the concept of 'best fit' motivating further discussion.

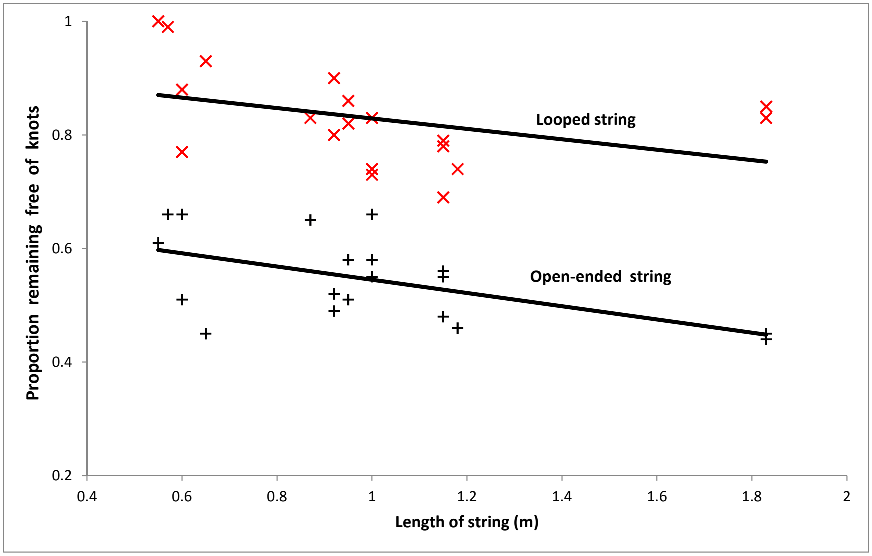

Figure 2 Data taken from Table 1 plotted to show dependence of freedom from knots on length of string and its state (open-ended or looped)

- *Do the data support the prediction of SAW theory?*
  This moves beyond the simple version of Murphy's Law to state that the probability of freedom from knots decreases *exponentially* with length according to Equation 1. Assessing whether this is true opens the way to discussion of more advanced issues, such as the use of proportions as a surrogate for probability, and the use of a (logarithmic) transformation to allow a best-fit line to be fitted to the data.
- *Do the data for looped string support the Loop Conjecture?*
  In its most basic form, this states simply that looped string is less likely to form a knot than open-ended string of the same length. Again, this requires interpretation of graphs like Figure 2, where the looped string data lie *above* the open-ended data. It also motivates discussion about the extent to which the two data sets can clearly be distinguished from one another, despite the ‘noise’.
- *How might the above questions be answered more quantitatively?*
  Is looking at data tables and plotting graphs enough ? This leads naturally to discussions of more advanced concepts such as regression lines as providing a ‘best fit’, the possibility that the findings are merely fluke results, and the use of statistical methods for investigating this.

Quantitative analysis

To illustrate the kind of quantitative analysis that can be performed on the data, we use the Coundon Court results, focusing on the issue of whether the results confirm the quantitative predictions of SAW theory in general, and the Loop Conjecture in particular.

1. As SAW theory predicts that the probability of freedom from knots follows the exponential dependence on string length given by Equation 1, we apply a logarithmic transformation to the knot-free proportions for both the open-ended and looped strings; this results in the plot shown in Figure 3.
2. The best-fit lines shown in Figure 3 were determined by linear regression to be:
   Open-ended string:
   $\ln(\text{proportion free of knots}) = -0.40 - 0.22L$
   Looped string:
   $\ln(\text{proportion free of knots}) = -0.09 - 0.10L$
3. The Loop Conjecture predicts that the act of looping reduces the value of α in Equation 1 by at least a factor of 2, so that $\beta \leq \alpha/2$. The data give regression slopes of $\alpha = -0.22$ and $\beta = -0.10$, so $\beta = \alpha/2.2$, which agrees with the theoretical prediction. A theoretical best-fit

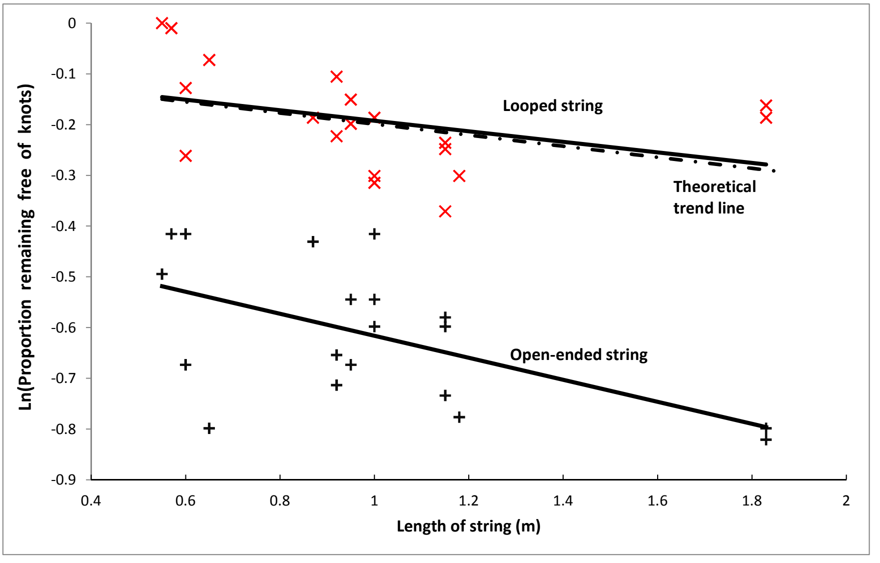

Figure 3 Logarithmically transformed data taken from Table 1

trend line is also shown in Figure 3; this is based on the Loop Conjecture's claim that the gradient of the looped string regression line should not exceed half the gradient for the open-ended string. As can be seen, there is a good fit with the prediction of the Loop Conjecture.

4 The visibly 'noisy' nature of the data sets is reflected by the correlation coefficients for the relationship between length and knot-free probability. While perfect (inverse) correlation would lead to correlation coefficients of −1, the values for the open-ended and looped strings are substantially different from this, at −0.56 and −0.38 respectively.

5 As there are only 19 data points, the question arises as to whether the apparent support for the theoretical predictions is actually just a fluke resulting from insufficient evidence. The conventional way to assess this is to examine the statistical significance of the two correlation coefficients, and calculate their *P* values; that is, the probabilities of getting at least as impressive a degree of correlation by chance alone. By convention, if a *P* value does not exceed 0.05, the result is deemed 'statistically significant'. Calculation of the significance of *P* values for correlation coefficients can be done via many online sites (technically, the so-called one-sided or directional hypothesis *P* value is the most appropriate). For the Coundon Court data, the correlation coefficients in both open-ended and looped cases is statistically significant ($P=0.01$ and 0.05 respectively).

Suggestions for further research

The nationwide project focused on investigating the principal predictions of SAW theory, and specifically the Loop Conjecture. Replications of the findings reported here could form the basis of further projects. For teachers wanting to explore as-yet uncharted territory with their students, other questions remain to be answered:

- **Extending the Loop Conjecture**
  The time-honoured way of preventing knots in rope has been to coil it many times, and then fix the ends. The Loop Conjecture implies, however, that forming just one coil and connecting the ends has a major impact on knot prevention, by halving the effective linear length available for knotting. Even so, as each additional coiling reduces the linear length by another factor of 2, the conjecture leads to the prediction that cord coiled into n loops will lead to a change in the bounds for the exponent in Equation 2 to $0<\beta<\alpha/2n$, where $n=1,2,3,\ldots$ This prediction of rapidly improving knot-free probability with each extra coil could be tested using the protocol used for the case of $n=1$, described in this article.

- **The effect of different materials**
  While knot theory shows that the risk of knot formation increases with cord length, other factors must also play a role, notably thickness of the cord, and its stiffness. For example, the flex making up headphone flex is substantially thicker and stiffer than the parcel string used in the experiment described here. One would thus expect that, length for length, headphone flex would be even less prone to knotting than parcel string. This could be investigated experimentally using lengths of flex instead of parcel cord.
- **The effects of thickness and stiffness**
  More advanced students could investigate the *relative* importance of cord thickness and stiffness for knotting risk. Clearly, thicker cord is also stiffer, despite being made of the same material. On the other hand, two cords of identical thickness can have different stiffness as a result of being made from materials of different inherent stiffness. One can separate the two effects via the concept of flexural rigidity, which for cord-like material is given by the formula $(\pi/64)YD^4$, where Y is Young's modulus, which varies from material to material, and D is the diameter of the cord. The relative importance of stiffness and thickness on knotting probability thus requires two separate experiments to be carried out: one with cords of identical thickness but different Young's modulus (in other words, different material), the other involving identical materials but different thickness.

Conclusion

The power and scope of mathematics and science is often conveyed via advances in fields far removed from everyday life, such as cosmology or particle physics. The notion that something as familiar yet seemingly 'lawless' as the knotting of headphone flex can also be amenable to mathematical and experimental investigation is likely to surprise many students. In this article, I have outlined the theory and experimental study of this phenomenon, which is of importance in biology and polymer science. I hope this will encourage both teachers and students to extend the investigations described here, and contribute to the understanding of phenomena that lie at the cutting edge of mathematical science.

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Robert Matthews is Visiting Reader in Science at the Department of Engineering and Applied Science, Aston University, Birmingham. He has made an extensive study of the history and scientific basis of Murphy's Law and other urban myths. Email: rajm@physics.org